1. Axioms of Set Theory

Axioms of Zermelo-Fraenkel

1.1. ***Axiom of Extensionality.*** *If X and Y have the same elements, then $X = Y$.*

1.2. ***Axiom of Pairing.*** *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

1.3. ***Axiom Schema of Separation.*** *If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.*

1.4. ***Axiom of Union.*** *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

1.5. ***Axiom of Power Set.*** *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

1.6. ***Axiom of Infinity.*** *There exists an infinite set.*

1.7. ***Axiom Schema of Replacement.*** *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*

1.8. ***Axiom of Regularity.*** *Every nonempty set has an $\in$-minimal element.*

1.9. ***Axiom of Choice.*** *Every family of nonempty sets has a choice function.*

The theory with axioms 1.1–1.8 is the Zermelo-Fraenkel axiomatic set theory ZF; ZFC denotes the theory ZF with the Axiom of Choice.

Why Axiomatic Set Theory?

Intuitively, a set is a collection of all elements that satisfy a certain given property. In other words, we might be tempted to postulate the following rule of formation for sets.

1.10. *Axiom Schema of Comprehension* (*false*). *If P is a property, then there exists a set $Y = \{x : P(x)\}$.*

This principle, however, is false:

1.11. Russell's Paradox. Consider the set S whose elements are all those (and only those) sets that are not members of themselves: $S = \{X : X \notin X\}$. Question: Does S belong to S? If S belongs to S, then S is not a member of itself, and so $S \notin S$. On the other hand, if $S \notin S$, then S belongs to S. In either case, we have a contradiction.

Thus we must conclude that

$$\{X : X \notin X\}$$

is not a set, and we must revise the intuitive notion of a set.

The safe way to eliminate paradoxes of this type is to abandon the Schema of Comprehension and keep its weak version, the *Schema of Separation*:

If P is a property, then for any X there exists a set $Y = \{x \in X : P(x)\}$.

Once we give up the full Comprehension Schema, Russell's Paradox is no longer a threat; moreover, it provides this useful information: The set of all sets does not exist. (Otherwise, apply the Separation Schema to the property $x \notin x$.)

In other words, it is the concept of the set of all sets that is paradoxical, not the idea of comprehension itself.

Replacing full Comprehension by Separation presents us with a new problem. The Separation Axioms are too weak to develop set theory with its usual operations and constructions. Notably, these axioms are not sufficient to prove that, e.g., the union $X \cup Y$ of two sets exists, or to define the notion of a real number.

Thus we have to add further construction principles that postulate the existence of sets obtained from other sets by means of certain operations.

The axioms of ZFC are generally accepted as a correct formalization of those principles that mathematicians apply when dealing with sets.

Language of Set Theory, Formulas

The Axiom Schema of Separation as formulated above uses the vague notion of a *property*. To give the axioms a precise form, we develop axiomatic set theory in the framework of the first order predicate calculus. Apart from the equality predicate $=$, the language of set theory consists of the binary predicate $\in$, the *membership relation.*