

# OUTLINE OF THE INTRODUCTION TO THE DIGITAL DIFFERENTIAL AND INTEGRAL CALCULUS

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## ABSTRACT

In this paper I go further from the digital continuous axiomatic Euclidean geometry ([8]) and introduce the basic definitions and derive the basic familiar properties of the differential and integral calculus without the use of the infinite, within finite sets only. No axioms are required in this only successfully chosen definitions. I call it the digital differential and integral calculus. Such mathematics is probably the old unfulfilled hitherto dream of the mathematicians since many centuries. Strictly speaking it is not equivalent to the classical differential and integral calculus which makes use of the infinite (countable and uncountable) and limits. Nevertheless for all practical reasons in the physical and social sciences it gives all the well known applications with a finite ontology which is directly realizable both in the physical ontology of atomic matter or digital ontology of operating systems of computers. Such a digital calculus has aspects simpler than the classical “analogue” calculus which often has a complexity irrelevant to the physical reality. It can become also more complicated than the classical calculus when more than 2 resolutions are utilized, but this complexity is directly relevant to the physical reality. The digital differential and integral calculus is of great value for the applied physical and social sciences as its ontology is directly corresponding to the ontology of computers. It is also a new method of teaching mathematics where there is integrity with what we say, write, see, and think. In this short outline of the basic digital differential and integral calculus, we include on purpose almost only the basic propositions that are almost identical with the corresponding of the classical calculus for reasons of familiarity with their proofs.

**Key words:** Digital mathematics, Calculus

MSC : 00A05

## 0. INTRODUCTION

Changing our concept of physical material, space and time continuum so as to utilize only finite points, numbers and sets, means that we change also our perception our usual mental images and beliefs about the reality. This project is under the next philosophical principles

- 1) In the human consciousness we have the experience of the infinite.**
- 2) But the ontology of the physical material world is finite.**
- 3) Therefore mathematical models in their ontology should contain only finite entities and should not involve the infinite.**
- 4) Strange as it may seem, the digital mathematics are the really deep mathematics of the physical world, while the classical mathematics of the infinite ("analogue" mathematics) are a “distant” phenomenology, convenient in older centuries, but not the true ontology.**

This paper is part of larger project which is creating again the basics of mathematics and its ontology with new definitions that do not involve the infinite at all.

Our perception and experience of the reality, depends on the system of beliefs that we have. In mathematics, the system of spiritual beliefs is nothing else than the axioms of the axiomatic systems that we accept. The rest is the work of reasoning and acting.

**Quote: "It is not the world we experience but our perception of the world"**

Nevertheless it is not wise to include in our perception of the material world and its ontology anything else than the finite, otherwise we will be lead in trying to prove basic facts with unsurpassed difficulties as the classical mathematics has already encounter , **The abstraction of the infinite is phenomenological** and it seems sweet at the beginning as it reduces some complexity, in the definitions, but later on it turns out to be bitter, as it traps the mathematical minds in to a vast complexity irrelevant to real life applications. Or to put it a more easy way, we already know the advantages of using the infinite but let us learn more about the advantages of using only the finite, for our perception, modelling and reasoning about empty space and physical reality. This is not only valuable for the applied sciences, through the computers but is also very valuable in creating a more perfect and realistic education of mathematics for the young people. H. Poincare used to say that mathematics and geometry is the art of correct reasoning over not-corresponding and incorrect figures. With the digital mathematics this is corrected. **The new digital continuums create a new integrity between what we see with our senses, what we think and write and what we act in scientific applications.**

The continuum with infinite many points creates an overwhelming complexity which is very often irrelevant to the complexity of physical matter. The emergence of the irrational numbers is an elementary example that all are familiar But there are less known difficult problems like the 3<sup>rd</sup> Hilbert problem (see [3] Boltianskii V. (1978)“). In the 3<sup>rd</sup> Hilbert problem it has been proved that two solid figures that are of equal volume are not always decomposable in to an equal finite number of congruent sub-solids! Given that equal material solids consists essentially from the physical point of view from an equal number of sub-solids (atoms) that are congruent, this is highly non-intuitive! There are also more complications with the infinite like the Banach-Tarski paradox (see [2] Banach, Stefan; Tarski, Alfred (1924)) which is essentially pure magic or miracles making! In other words it has been proved that starting from a solid sphere S of radius r, we can decompose it to a finite number n of pieces, and then re-arrange some of them with isometric motions create an equal sphere S1 of radius again r and by rearranging the rest with isometric motions create a second solid Sphere S2 again of radius r! In other words like magician and with seemingly elementary operations we may produce from a ball two equal balls without tricks or “cheating”. Thus no conservation of mass or energy!. Obviously such a model of the physical 3-dimensional space of physical matter like the classical Euclidean geometry is far away from the usual physical material reality! I have nothing against miracles, but it is challenging to define a space , time and motion that behaves as we are used to know. In the model of the digital 3-dimesional space, where such balls have only finite many points such “miracles” are not possible!

The current digital version of the differential and integral calculus is based on the atomic structure of matter as hypothesized 2,000 years ago by the ancient Greek philosopher Democritus and which has developed in the modern the atomic physics. Also the role of computers and their digital world is important as it shows that space, time, motion, images, sound etc can have finite digital ontology and still can create the continuum as a phenomenology of perception.

The famous physicist E. Schrodinger in his book ([12] *E. Schroedinger. Science and Humanism Cambridge University press 1961*) mentions that the continuum as we define it with the “analogue” mathematics involving the infinite is problematic and paradoxical, therefore needs re-creation and re-definition. It is exactly what we do here with the digital differential and integral calculus.

We enumerate some great advantages of the digital differential and integral calculus compared to the classical calculus with the infinite.

- 1) The digital continuity and smoothness (derivative) allows for a variable in scale of magnitude and resolution such concept and not absolute as in analogue classical mathematics. A curve may be smooth (differentiable) at the visible scale but non-smooth at finer scales and vice versa. This is not possible with classical definitions.
- 2) Corresponding to the concept of infinite of classical mathematics in digital mathematics there is the concept seemingly infinite and seemingly infinitesimal at its various orders, which is still finite. Thus many unprovable results in classical mathematics become provable in digital mathematics. This also resurrects the 17<sup>th</sup> and 18<sup>th</sup> century mathematical arguments in Calculus and mathematical physics that treated the “infinitesimals” as separate entities in the derivatives.
- 3) Many unsurpassed difficulties in proving desirable results in the infinite dimensional functional spaces of mathematical analysis disappear and allow for new powerful theorems because the seemingly infinite is still finite.
- 4) Integration is defined as finite (although seemingly infinite) weighted sum of the volumes of the points at some precision level, exactly as Archimedes was measuring and integrating volumes with water or sand. Contrary to classical mathematics any computably bounded function is integrable (see proposition 3.6 )
- 5) Therefore, there are vast advantages compared to classical analogue calculus. The digital differential and integral calculus is a global revolution in the ontology of mathematics in teaching and applying them comparable with the revolution of digital technology of sounds, images . motion, etc compared to the classical analogue such technologies.
- 6) There are although “disadvantages” too , in the sense that if we do not restrict to a digital calculus relative 4 precision levels but include many more and grades of differentiability and integrability then the overall calculus will become much more complicated than the classical calculus.

In this short outline of the basic digital differential and integral calculus, we include on purpose only the basic propositions that are almost identical with the corresponding of the classical calculus for reasons of familiarity with their proofs. An exception is the proposition 3.6 which has an almost obvious proof.

## **1. THE DEFINITION OF THE DIGITAL REAL NUMBERS**

### THE MULTI-PRECISION DECIMAL DIGITAL REAL NUMBERS $R(m,n,p,q)$

## Rules for phantasy and drawing of figures.

*As initially we considered a system of digital real numbers  $R(m,n,p,q)$  we consider the points of  $P(m)$ ,  $P(n)$  as visible in the figures while the points of  $P(p)$  as invisible pixels, and those of  $P(q)$  as invisible atoms. Therefore, even the points and seemingly infinitesimals that will be defined below, of  $P(n)$  relative to  $P(m)$  are considered as visible. This is in accordance with the habit in classical mathematics to make the points visible, although they claim that they have zero size.*

- a) The **rational numbers**  $Q$ , as we know them, do involve the infinite, as they are infinite many, and are created with the goal in mind that **proportions**  $k/l$  of natural numbers  $k,l$  exist as numbers and are unique. The cost of course is that when we represent them with decimal representation they may have infinite many but with finite period of repetition decimal digits.
- b) The **classical real numbers**  $R$ , as we know them, do involve the infinite, as they are infinite many, and are created with the goal in mind that **proportions** of linear segments of Euclidean geometry, exist as numbers and are unique (**Eudoxus** theory of proportions). The cost of course finally is that when we represent them with decimal representation they may have infinite many arbitrary different decimal digits without any repetition.
- c) But in the physical or digital mathematical world, such costs are not acceptable. The infinite is not accepted in the ontology of digital mathematics (only in the subjective experience of the consciousness of the scientist). Therefore in the **multi-precision digital real numbers**, proportions are handled in different way, with priority in the Pythagorean-Democritus idea of the **creation of all numbers from an integral number of elementary units**, almost exactly as in the physical world matter is made from atoms (here the precision level of numbers in decimal representation) and the definitions are different and more economic in the ontological complexity.

We will choose for all practical applications of the digital real numbers to the digital Euclidean geometry and digital differential and integral calculus, the concept of a system of digital decimal real numbers with three precision levels, lower, low and a high.

**Definition 1.1** The definition of a PRECISION LEVEL  $P(n,m)$  where  $n, m$  are natural numbers, is that it is the set of all real numbers that in the decimal representation have not more than  $n$  decimal digits for the integer part and not more than  $m$  digits for the decimal part. Usually we take  $m=n$ . In other words as sets of real numbers it is a nested system of lattices each one based on units of power of 10, and as union a lattice of rational numbers with finite many decimal digits. We could utilize other bases than 10 e.g. 2 or 3 etc, but for the sake of familiarity with the base 10 and the 10 fingers of our hands we leave it as it is.

## THE DEFINITION 1.2 OF THE DIGITAL REAL NUMBERS $R(m,n,p,q)$

**We assume at least four precision levels for an axiomatic decimal system of digital real numbers**

Whenever we refer to a real number  $x$  of a (minimal in precision levels) system of real numbers  $R(m,n,p,q)$ , we will always mean that  $x$  belongs to the precision level  $P(m)$  and that the system  $R(m,n,p,q)$  has at least four precision levels with the current axioms.

Whenever we write an equality relation  $=_m$  we must specify in what precision level it is considered. The default precision level that an equality of numbers is considered to hold, is the precision level  $P(m)$ .

### **Some of the Linearly ordered Field operations**

The field operations in a precision level are defined in the usual way, from the decimal representation of the numbers. This would be an independent definition, not involving the infinite. Also equality of two numbers with finite decimal digits should be always specified to what precision level. E.g. if we are talking about equality in  $P(m)$  we should symbolize it by  $=_m$ , while if talking about equality in  $P(q)$  we should symbolize it by  $=_q$ . If we want to define these operations from those of the real numbers with infinite many decimal digits, then we will need the truncation function  $[a]_x$  of a real number  $a$ , in the Precision level  $P(x)$ .

Then the operations e.g. in  $P(m)$  with values in  $P(n)$   $m < n$  would be

$$[a]_m + [b]_m =_n [a+b]_n \quad (\text{eq. 3})$$

$$[a]_m * [b]_m =_n [a*b]_n \quad (\text{eq. 4})$$

$$([a]_m)^{-1} =_n [a^{-1}]_n \quad (\text{eq. 5})$$

(Although, the latter definition of inverse seems to give a unique number in  $P(n)$ , there may not be any number in  $P(n)$  or not only one number in  $P(n)$ , so that if multiplied with  $[a]_m$  it will give 1. E.g. for  $m=2$ , and  $n=5$ , the inverse of 3, as  $([3]_m)^{-1} =_n [1/3]_n = 0.33333$  is such that still  $0.33333 * 3 \neq_n 1$ ).

Nevertheless here we will not involve the infinite and the classical real numbers, and we take the operation of digital real numbers from the standard operations of them as numbers with finite digital decimal representation and truncation by rounding.

Such a system of double or triple precision digital real numbers, has closure of the linearly ordered field operations only in a specific local way. That is If  $a, b$  belong to the Local Lower precision, then  $a+b, a*b, -a, a^{-1}$  belong to the Low precision level, and the properties of the linearly ordered commutative field hold: (here the equality is always in  $P(n)$ , this it is mean the  $=_n$ ).

1) if  $a, b, c$  belong to  $P(m)$  then  $(a+b), (b+c), (a+b)+c, a+(b+c)$  belong in  $P(n)$  and  $(a+b)+c = a+(b+c)$  for all  $a, b$  and  $c$  in  $P(n)$ .

2) There is a digital number 0 in  $P(n)$  such that

2.1)  $a+0 = a$ , for all  $a$  in  $P(n)$ .

2.2) For every  $a$  in  $P(m)$  there is some  $b$  in  $P(n)$  such that

$a+b=0$ . Such  $a, b$  is symbolized also by  $-a$ , and it is unique in  $P(n)$ .

3) if  $a, b$ , belong to  $P(m)$  then  $(a+b), (b+a)$ , belong in  $P(n)$  and

$a+b = b+a$

4) if  $a, b, c$  belong to  $P(m)$  then  $(a*b), (b*c), (a*b)*c, a*(b*c)$  belong in  $P(n)$  and

$(a*b)*c = a*(b*c)$ .

5) There is a digital number 1 in  $P(n)$  not equal to 0 in  $P(n)$ , such that

5.1)  $a*1 = a$ , for all  $a$  in  $P(n)$ .

5.2) For every  $a$  in  $P(m)$  not equal to 0, there may be one or none or not only one  $b$  in  $P(n)$  such that  $a*b = 1$ . Such  $b$  is symbolized also by  $1/a$ , and it may not exist or it may not be unique in  $P(n)$ .

6) if  $a, b$ , belong to  $P(m)$  then  $(a*b), (b*a)$ , belong in  $P(n)$  and

$$a*b=b*a$$

7) if  $a, b, c$  belong to  $P(n)$  then  $(b+c), (a*b), (a*c), a*(b+c), a*b+a*c$ , belong in  $P(n)$  and  $a*(b+c)=a*b+a*c$

Which numbers are positive and which negative and the linear order of digital numbers is precision levels  $P(m), P(n)$ , etc is something known from the definition of precision levels in the theory of classical real numbers in digital representation.

If we denote by  $PP(m)$  the positive numbers of  $P(m)$  and  $PP(n)$  the positive numbers of  $P(n)$  then

8) For all  $a$  in  $PP(m)$ , one and only one of the following 3 is true

8.1)  $a=0$

8.2)  $a$  is in  $PP(m)$

8.3)  $-a$  is in  $PP(m)$  ( $-a$  is the element such that  $a+(-a)=0$ )

9) If  $a, b$  are in  $PP(m)$ , then  $a+b$  is in  $PP(n)$

10) If  $a, b$  are in  $PP(m)$ , then  $a*b$  is in  $PP(n)$

It holds for the inequality  $a>b$  if and only if  $a-b$  is in  $PP(n)$

$a<b$  if  $b>a$

$a<=b$  if  $a<b$  or  $a=b$

$a>=b$  if  $a>b$  or  $a=b$

and similar for  $PP(n)$ .

**Similar properties as the ones from  $P(m)$  to  $P(n)$  hold if we substitute  $n$  with  $m$ , and  $m$  with  $p, q$ .**

**For the  $R(m,n)$  the integers of  $P(m)$  are also called computable finite or countable finite, while those of  $P(n)$  are unaccountable finite or non-computable finite or also seemingly infinite relative to  $P(m)$ .**

Also, the Archimedean property holds only recursively in respect e.g. to the local lower precision level  $P(m)$ .

In other words, if  $a, b, a<b$  belong to the precision level  $P(m)$  then there is  $k$  integer in the precision level  $P(n)$  such that  $a*k>b$ . And similarly for the precision levels  $P(n)$  and  $P(p), P(q)$ .

The corresponding to the Eudoxus-Dedekind completeness in the digital real numbers also is relative to the three precision levels is simply that in the precision levels all possible combination of digits are included and not any decimal number of  $P(m)$  or  $P(n)$  is missing. Still this gives

**THE SUPREMUM COMPLETENESS PROPERTY OF THE DIGITAL REAL NUMBERS.**

From this completeness we deduce the supremum property of upper bounded sets (and infimum property of lower bounded sets) in the  $P(m)$  (but also  $P(n)$ ) precision levels. This is because

*in well ordered sets holds the supremum property of upper bounded sets. Here lower bounded sets have also the infimum property and this holds for any resolution of the digital real numbers*

**Mutual inequalities of the precision levels (AXIOMS OF SEEMINGLY (m,n) -INFINITE OR (m,n)-UNCOUNTABLE OR NONO-COMPUTABLE FINITE AMONG RESOLUTIONS and seemingly finite or visibly finite or bounded or computable finite numbers. )**

We impose also axioms for the sufficiently large size of the high precision level relative to the other two, and the sufficient large size of the low precision level relative to the local lower precision level. That is for the mutual relations of the integers m, n, p, q.

It may seem that these differences of the resolution or the precision levels are very severe and of large in between distance, and not really necessary. It may be so, as the future may show. But for the time being we fell safe to postulate such big differences.

There are definitions modeled after the definitions of **inaccessible cardinals** in classical mathematics. Here we give a weaker alternative definitions with **weaker concepts of seemingly infinite** that would correspond to that of inaccessible cardinals. In other words **we do not include the operation of power.**

We may conceive the **countable finite** as a finite computable by a computational power of some computer, and **unaccountable finite as the finite not computable** by a type of a computer

**Transcendental Orders of (m,n) seemingly infinite, as in classical mathematics transcendental orders of ordinal numbers are also definable. E.g. if a, b are (m,n)=seemingly infinite then a is transcendental larger than b, in symbols  $a \gg b$  iff  $b/a =_m 0$  in  $P(m)$ .**

**And similarly transcendental orders of seemingly infinitesimals. E.g. if a, b are (m,n)=seemingly infinitesimals then a is transcendental smaller than b, in symbols  $a \ll b$  iff  $a/b =_m 0$  in  $P(m)$ .**

We may compare them with the small  $o()$  and big  $O()$  definitions of the classical mathematics, but they are different as the latter involve the countable infinite, while former here involve only finite sets of numbers.

**9) REQUIREMENTS OF THE SEEMINGLY INFINITE** *If we repeat the operations of addition and multiplication of the linearly ordered commutative field starting from numbers of the precision level  $P(m)$ , so many times as the numbers of the local lower precision level  $P(m)$ , then the results are still inside the low precision level  $P(n)$ . In symbols if by  $|P(m)|$  we denote the cardinality of  $P(m)$ , then*

*$|P(m)| \cdot (10^m)$ , and  $(10^m)^{|P(m)|} \leq 10^n$ . Similarly for the pair (m,q). We may express it by saying that the  $10^n$  is seemingly infinite or unaccountable finite compared to  $10^m$ , or that the numbers less than  $10^n$  are countable or computable finite. If we include besides the addition and multiplication the power operation too, then  $10^m$  is inaccessible seemingly infinite compared to  $10^m$  (a concept similar to inaccessible cardinal numbers in classical mathematics). Similarly for the precision levels  $P(p)$ ,  $P(q)$ .*

**10) REQUIREMENTS OF THE SEEMINGLY INFINITESIMALS** *The smallest*

magnitude in the low precision level  $P(n)$  in other words the  $10^{-n}$ , will appear as zero error in the low precision level  $P(m)$ , even after additive repetitions that are as large as the cardinal number of points of the lower precision level  $P(m)$  and multiplied also by any large number of  $P(m)$ . In symbols

$10^{-n} * |P(m)| * 10^m \leq 10^{-m}$ . Similarly for the pairs  $(n,p)$ ,  $(p,q)$ .

This may also be expressed by saying that the  $10^{-n}$  is seemingly infinitesimal compared to the  $10^{-m}$ . Other elements of  $P(n)$  symbolized by  $dx$  with  $|dx| < 10^{-m}$  with the same inequalities, that is  $|dx| * |P(m)| * 10^m \leq 10^{-m}$  are also seemingly infinitesimals, provided the next requirements are also met:

The seemingly infinitesimals  $dx$  of  $P(p)$  relative to  $P(m)$  (thus  $|dx| \leq 10^{-m}$ ) are by definition required to have properties that resemble the **ideals in ring theory** (see e.g. [15] VAN DER WAERDEN ALGEBRA Vol I, chapter 3, Springer 1970). More precisely what it is required to hold is that

If  $a, b$  are elements of  $P(m)$ , and  $dx, dy$  seemingly infinitesimals of  $P(p)$  relative to  $P(n)$  (thus  $|dx|, |dy| \leq 10^{-n}$ , thus relative to  $P(m)$  too) then the linear combination and product are still seemingly infinitesimals. In symbols  $adx + bdy$ , are seemingly infinitesimals of  $P(n)$  relative to  $P(m)$  and  $dx * dy$  is seemingly infinitesimal of  $P(q)$  relative to  $P(p)$  and thus relative to  $P(m)$  too.

We call this **the ideal-like property of the seemingly infinitesimals**.

One very important equation is of course that the digital real numbers is the union of the four precision levels.

$$R(m, n, p, q) = P(m) \cup P(n) \cup P(p) \cup P(q)$$

Two digital systems of Real numbers  $R(m,n,p,q)$ ,  $R(m',n',p',q')$  with  $m=m'$ ,  $n=n'$ ,  $p=p'$ ,  $q=q'$  and the above axioms are considered isomorphic.

## 2. THE DEFINITION OF THE DIGITAL FUNCTIONS, DIGITAL CONTINUITY AND DIGITAL DIFFERENTIABILITY.

A **digital real function** at 2 precision levels is a function in the ordinary set-theoretic sense, that sends elements of the digital real numbers to elements of the digital real numbers. It has to be defined so that **it respects the precision levels**. This is defined so that a **parallelogram diagram**, of the two functions, the restriction function and the rounding function **commute** in the sense of the theory of **categories**. Usually the standard way is to **define it for the highest resolution** and then extend the definition for the lower resolutions by the **rounding function**



(left for positive numbers and right for negative numbers). This process is called **natural rounding extension on lower resolutions**, and defines the rounded functions on the lower resolutions so that the arrow diagrams commute that  $[f(a)]_n = f_n([a]_n)$  if  $a, f(a)$  in  $P(q)$  and we define  $f$  on  $P(n)$  (The rounding of the image is the value of the rounded function on the rounded argument, so that rounding function and functions commute). **We only need to define the rounding for a pair of precision levels for differentiation and integration. Here for  $P(m)/P(n)$ . The  $f_m$  is the rounded function, and it is for all practical purposes the one only function observed. But it starts from a function  $f$  on  $P(n)$ . So for all digital function that we will consider, we will conceive them as double functions the finest**

**of:  $P(n) \rightarrow P(n)$  and the rounded,  $f: P(m) \rightarrow P(m)$ , and  $r$  is the restriction from  $P(n)$  to  $P(m)$  then a commutation of diagrams is the  $[(of([x]_m))]_{m=mf}$**

**In some situations (e.g. definition of continuity) we will assume that the digital function is defined in 3 precision levels  $oof: P(p) \rightarrow P(p)$  of:  $P(n) \rightarrow P(n)$  and the rounded,**

**$f: P(m) \rightarrow P(m)$ , and by the restriction from  $P(n)$  to  $P(m)$  and from  $P(n)$  to  $P(p)$  a commutation of diagrams holds:  $[(of([x]_m))]_{m=mf}$  and  $[(oof([x]_n))]_{n=of}$ .**

**And in some cases we will need all 4-precision levels**

For those that feel convenient to start with the classical mathematics with the infinite, and their functions, digital functions as above can be obtained by the rounding functions  $[ ]_m [ ]_n$  in the precision levels  $P(m), P(n)$ . E.g. starting with the classical exponential function  $g(x) = e^x$  to obtain a digital function in  $P(m), P(n)$ , we use the formulae  $oof(x) = [e^{[x]_p}]_p$ ,  $of(x) = [e^{[x]_n}]_n$  and  $f(x) = [e^{[x]_m}]_m$

## **DEFINITION 2.1**

*A digital real function defined on a closed interval  $f: [a, b]_m \rightarrow P(m)$ , of:  $[a, b]_n \rightarrow P(n)$ , oof:  $[a, b]_p \rightarrow P(p)$  is **(digitally)  $P(m)/P(n)/P(n)$  continuous at a point  $x$**  of its domain of definition  $[a, b]_m$  in  $P(m)$ , if and only if for every other point  $x'$  of the domain of definition  $[a, b]_n$  in  $P(n)$ , such that  $x, x'$  are of seemingly infinitesimally distance  $dx = x' - x$  (belongs to  $P(n)$ ), relative to  $P(m)$ , then also the  $dy = of(x') - of(x)$  is seemingly infinitesimal of  $P(n)$  relative to  $P(m)$ . It holds in particular:*

$$dy =_n dof(x) =_m dx =_m 0$$

*Similar definitions hold for  $P(m)/P(p)$ ,  $P(n)/P(p)$  and  $P(m)/P(q)$  continuity.*

We concentrate on functions of  $P(n)$  of  $R(m,n,p,q)$  but we may we not leave unused the precision levels  $P(p)$ ,  $P(q)$ . We mention also that the definitions can be also for the triples of precision levels  $P(m)$ - $P(n)$ - $P(p)$ ,  $P(n)$ - $P(p)$ - $P(q)$  as finer forms of continuity. If it is for all precision levels then it seems equivalent to the classical definitions.

If digital real function is digitally continuous at all points of its domain of definition it is called a **(digitally)  $P(m)/P(n)$  continuous digital real function**.

## DEFINITION 2.2

A digital real function defined on a closed interval  $f:[a,b]_m \rightarrow P(m)$ ,  $of: [a,b]_n \rightarrow P(n)$ ,  $oof: [a,b]_p \rightarrow P(p)$  is **(digitally)  $P(m)/P(n)/P(p)$  continuous at a point  $x$**  of its domain of definition  $[a,b]_m$  in  $P(m)$ , if and only if for every other point  $x'$  of the domain of definition  $[a,b]_p$  in  $P(p)$ , such that  $x, x'$  are of seemingly infinitesimally distance  $dx = x' - x$  (belongs to  $P(p)$ ), relative to  $P(m)$ , then also the  $dy = of(x') - of(x)$  is seemingly infinitesimal of  $P(n)$  relative to  $P(m)$ . It holds in particular:

$$0 =_m dy =_n dof(x) =_m dx =_m 0$$

Similar definitions hold for  $P(n)/P(p)/P(q)$ ,  $P(m)/P(p)/P(q)$  continuity etc.

**It would be nice if it is possible to derive also the digital  $P(m)/P(n)$  continuity as the standard continuity of topological space.** The next definition gives the best idea for such a topological space. A topological space is defined by its open sets (see e.g. [9] J.Munkress). But the open sets can also be definite by the limit points of sets too.

We consider the Cartesian product set  $P(m) \times P(n) = P(m) \times P(n)$ , where we define the disjoint union space  $P(m) + P(n)$  and we do not consider that a coarse point of  $P(m)$  contains fine points of  $P(n)$  but we treat them separately. Our topological space will be the  $Y = X + oX = P(m) + P(n)$ . Subsets  $A$  of  $Y$  can be split to  $A = oA + cA$ , where  $oA$  are the fine points of  $A$  in  $P(n)$  and  $cA$  are the coarse points of  $A$  in  $P(m)$ .

## DEFINITION 2.3

A point  $x$  of  $X = P(\text{low}) = P(m)$  or of  $oX = P(\text{high}) = P(n)$  is a limit point of a subset  $A$  of  $Y = P(\text{low}) + P(\text{high})$  (and  $oA$  is a subset of  $oX$ ), iff there is a positive seemingly infinitesimal  $de$  of  $P(\text{high})$  such that for any positive seemingly infinitesimal  $da$  of  $P(\text{high})$  less than  $de$ , there is a fine point  $y$  of  $oA$  such that  $|x - y| = da$ . We denote the set of fine points of  $P(\text{high})$  limit points of  $A$  by  $oL(A)$  and all coarse points of  $P(\text{low}) = P(m)$  by  $L(A)$ . We define as closure  $cl(A)$  of a subset  $A$  of  $Y$ , the  $cl(A) = A$  union  $Cl(A)$ . A set is open if its complement in  $Y$  is the closure of a set.

Notice that with the closure we add only coarse visible points not fine (possibly invisible) points. For this reason the closure operator has the idempotent low  $Cl(Cl(A))=Cl(A)$ . For the relations of limit points, closure, boundary, open sets etc see [9] J. Munkres. In addition  $Cl(A \cup B) = Cl(A) \cup Cl(B)$  and  $Cl(A \cap B) = Cl(A) \cap Cl(B)$ . We define that a  $x$  point of  $Y$  is seemingly in contact with the subset  $A$  of  $Y$  iff  $x$  belongs to  $A \cup Cl(A)$ . In other words either it belongs to the set or it is a limit point of it.

The concepts of boundary points and interior points are defined so as to have the usual properties as well as the concept of open set, base of open sets and base of neighbourhoods in  $Y$ . Similarly for connectedness. (See e.g. [9] J.Munkres)

The concept of topological **lowest visible or accountable or computable compactness** is defined in the usual way, where far the existence of finite sub-cover for any cover, we require, existence of **lowest visibly finite cardinality of a sub cover**. Similarly for the concept of **lower visible or computable or accountable compactness** or simply visibly compactness of a set of points. For a first outline of the Digital Calculus we will not proceed in these details.

The basic properties of continuity are:

- 1) Continuity is invariant by linear combinations
- 2) Continuity and product
- 3) Continuity and quotient
- 4) Composition of digital continuous functions are digital continuous
- 5) Bolzano theorem (after the supremum property of digital real numbers)
- 6) Mean value theorem.

### PROPOSITION 2.1 (CONTINUOUS COMPOSITE)

*Let two digital functions  $f: [\alpha, \beta]_m \rightarrow R(m, n)$ , with  $oof: [\alpha, \beta]_p \rightarrow P(p)$ ,  $of = [oof]_n$ ,  $f = [of]_m$  and  $h: [f(a), f(b)]_m \rightarrow R(m, n)$ , with  $ooh: [f(a), f(b)]_p \rightarrow P(p)$ ,  $oh = [ooh]_n$ ,  $h = [oh]$ , that the first is (digitally)  $P(m)/P(n)/P(p)$  continuous and the second  $P(m)/P(p)/P(q)$  continuous such that their composition  $f(h)(x): [\alpha, \beta]_m \rightarrow R(m, n)$  defined by  $oor = [oof(ooh([x]_p))]$  (and or,  $r$  defined in the obvious way), is also a digital function with values in  $P(m)$  (in other words its diagram commutes). Then this composition function is also a (digitally)  $P(m)/P(n)/P(q)$  continuous function in  $[a, b]_m$ .*

**Hint for a proof:** From the definition of the composite digital function  $oor$  on  $x$  of  $[a, b]_p$  if  $dx$  is a seemingly infinitesimal at  $x$  of  $P(p)$ , then from the  $P(m)/P(n)/P(p)$  continuity of  $ooh$  at  $x$  we get that the  $dy = {}_n ooh(dx)$  is a seemingly infinitesimal of  $P(n)$  relative to  $P(m)$ , and from the  $P(m)/P(n)$  continuity of the  $of$  we get that the  $of(dy)$  is a seemingly infinitesimal of  $P(n)$ , relative to  $P(m)$ . Thus the composite  $r$  is digitally  $P(m)/P(p)$  continuous. QED

## PROPOSITION 2.2 (CONTINUOUS LINEAR COMBINATIONS)

Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$

, and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(p)/P(q)$  continuous such that for any digital scalars  $a, b$  of  $P(m)$ , the functions  $af+bh, f^*h, 1/f$  are also digital functions on  $[a,b]_m$  with values in  $P(m)$ , then they are also (digitally)  $P(m)/P(n)/P(q)$  continuous functions.

**Hint for a proof:** From the  $P(m)/P(p)/P(q)$  continuity of the  $f$  and  $h$  we get that for  $dx$  seemingly infinitesimals of  $P(q)$ , the  $df(x), dh(x)$  are seemingly infinitesimals of  $P(p)$  and from the ideal-like property of the  $P(p)$  seemingly infinitesimals (see definition of digital real numbers 10) ) the  $adf(x)+bdh(x)$  is a seemingly infinitesimal of  $P(n)$  relative to  $P(m)$ , thus the linear combination is  $P(m)/P(n)/P(q)$  digital continuous. QED

## PROPOSITION 2.3 (CONTINUOUS PRODUCT)

Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$

, and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(p)/P(q)$  continuous such that for any digital scalars  $a, b$  of  $P(m)$ , the functions  $f^*h$ , is also digital functions on  $[a,b]_m$  with values in  $P(m)$ , then they are also (digitally)  $P(m)/P(n)/P(q)$  continuous functions.

**Hint for a proof:** From the  $P(m)/P(p)/P(q)$  continuity of the  $f$  and  $h$  we get that for  $dx$  seemingly infinitesimals of  $P(q)$ , the  $df(x), dh(x)$  are seemingly infinitesimals of  $P(p)$  and from the ideal-like property of the  $P(p)$  seemingly infinitesimals (see definition of digital real numbers 10) ) the  $df(x)*dh(x)$  is a seemingly infinitesimal of  $P(q)$  relative to  $P(m)$ . Then the  $df(x)h(x) =_p f(x+dx)h(x+dx) - f(x)h(x) =_p (f(x)+df(x))(h(x)+dh(x)) - f(x)h(x) =_p$  by multiplying out we get a linear combination of seemingly infinitesimals of  $P(p)$  and  $P(q)$  that by the ideal-like property of the seemingly infinitesimals are also seemingly infinitesimals of  $P(n)$  relative to  $P(m)$ . Thus the product is  $P(m)/P(n)/P(q)$  digital continuous. QED

## PROPOSITION 2.4 (CONTINUOUS INVERSE)

Let a digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$

, that is (digitally)  $P(m)/P(p)/P(q)$  continuous such that the functions  $1/f$  is also definable digital functions on  $[a,b]_m$  with values in  $P(m)$ , then it is also (digitally)  $P(m)/P(n)/P(q)$  continuous function.

**Hint for a proof:** From the  $P(m)/P(p)/P(q)$  continuity of the  $f$  we get that for  $dx$  seemingly infinitesimals of  $P(q)$ , the  $df(x)$ , is seemingly infinitesimals of  $P(p)$ . The  $d(1/f(x)) =_p (f(x+dx) - f(x))/f(x)*(f(x+dx) =_p (df(x))/f(x)*(df(x)+f(x))$ . The denominator is a computable finite number

and non-seemingly infinitesimal of  $P(m)$ , while the numerator is a seemingly infinitesimal of  $P(p)$ . From the ideal-like properties of the seemingly infinitesimals we deduce that the ratio is a seemingly infinitesimal of  $P(n)$ . Thus the inverse is  $P(m)/P(n)/P(q)$  digital continuous. QED

**PROPOSITION 2.5 (BOLZANO)**

*Let a digital (digitally) continuous functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$   $f:P(m) \rightarrow P(m)$ , defined in a finite interval  $[a,b]_m$  of  $P(m)$  such that  $f(a), f(b)$  have opposite signs, that is  $f(a)f(b) <_m 0$ , (e.g. assume  $f(a) <_m 0$ ) then there is at least one point  $c$  in the open interval  $(a,b)_m$ , such that for its next higher point  $c'$  in  $[a,b]_m$  holds  $f(c) <_m 0$  and  $f(c') >_m 0$*

**Hint for a proof:** We apply the supremum completeness property for upper bounded sets of the digital real numbers at the  $P(m)$  precision level for the set  $A = \{x / a <_m x <= b \text{ that the } f \text{ is negative in the } [a,x] \}$ . QED

**PROPOSITION 2.6 (MAXIMUM)**

*Let a digital (digitally) continuous functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$   $f:P(m) \rightarrow P(m)$ , defined in a finite interval  $[a,b]_m$  of  $P(m)$ , then it attains its maximum in  $[a,b]_m$ , in other words there is a number  $y$  in  $[a,b]_m$  in  $P(m)$ , such that  $f(x) <= f(y)$  for all  $x$  in  $[a,b]_m$  in  $P(m)$ .*

**Hint for a proof:** We apply the supremum property of the digital real numbers at the  $P(m)$  precision level for the set  $A = f([a,b])$  in  $P(m)$ . As  $A$  is a finite set it has a maximum element.

**DEFINITION 2.3**

*A digital real function defined on a closed interval  $f:[a,b]_m \rightarrow P(m)$ ,  $of: [a,b]_n \rightarrow P(n)$ , is (digitally) is  $P(m)/P(n)/P(n)$  differentiable at a point  $a$  of its domain of definition  $[a,b]_m$  in  $P(m)$ , if for every other point  $x'$  of its domain of definition  $[a,b]_n$  in  $P(n)$ , such that the distance of  $a$  and  $x'$  is seemingly infinitesimal belonging in  $P(n)$  and relative to  $P(m)$  with  $dx =_n x' - a$ , then  $dy =_n f(x') - f(a)$  is a seemingly infinitesimal relative to  $P(m)$ , belonging to  $P(n)$  and the ratio  $dy/dx =_m (f(x') - f(a)) / (x' - a)$  is always the same as number  $c$  of  $P(m)$ , independent from the choice of  $x'$  which is called the derivative of  $f$  at  $a$ ,  $c =_m df(x)/dx|_a$ , while the  $c \cdot dy/dx =_n c \cdot (f(x') - f(a)) / (x' - a)$  is a seemingly infinitesimal relative to  $P(m)$  and belonging to  $P(n)$ .*

Notice that when change seemingly infinitesimals  $dx$ , the  $dy/dx$  may change as number of  $P(n)$ , but remains constant as number of  $P(m)$ .

Similarly we may define differentiation by the pairs of precision levels  $P(m)-P(p)$ , and  $P(m)-P(q)$ .

## DEFINITION 2.4

A digital real function defined on a closed interval  $f : [a, b]_m \rightarrow P(m)$ , of:  $[a, b]_n \rightarrow P(n)$ , is (digitally)  $P(m)/P(n)/P(p)$  differentiable at a point  $a$  of its domain of definition  $[a, b]_m$  in  $P(m)$ , if for every other point  $x'$  of its domain of definition  $[a, b]_n$  in  $P(n)$ , such that the distance of  $a$  and  $x'$  is seemingly infinitesimal (that is in  $P(p)$  and relative to  $P(n)$  and  $P(m)$ ) with  $dx = x' - a$ , then  $dy = f(x') - f(a)$  is a seemingly infinitesimal relative to  $P(m)$ , belonging to  $P(p)$  and the ratio  $dy/dx = (f(x') - f(a))/(x' - a)$  is always the same as number  $c$  of  $P(m)$ , independent from the choice of  $x'$  which is called the derivative of  $f$  at  $a$ ,  $c = \frac{df(x)}{dx}|_a$ , while the  $c - dy/dx = c - (f(x') - f(a))/(x' - a)$  is a seemingly infinitesimal relative to  $P(m)$  and belonging to  $P(n)$ .

Notice that when change seemingly infinitesimals  $dx$ , the  $dy/dx$  may change as number of  $P(n)$ , but remains constant as number of  $P(m)$ .

Similarly we may define differentiation by the pairs of precision levels  $P(m)-P(p)$ , and  $P(m)-P(q)$ .

The basic properties of differentiability are

- 1) Chain Rule
- 2) Linearity
- 3) Product or Leibniz rule
- 4) Quotient rule

## PROPOSITION 2.7 (CHAIN RULE)

Let two digital functions  $f : [a, b]_m \rightarrow R(m, n)$ , with  $of : [a, b]_n \rightarrow P(n)$ ,  $f = [of]$ , and  $h : [f(a), f(b)]_m \rightarrow R(m, n)$ , with  $oh : [f(a), f(b)]_n \rightarrow P(n)$ ,  $h = [oh]$ , that are the first (digitally)  $P(m)/P(n)/P(p)$  differentiable at  $a$  (in  $P(m)$ ) and the second  $P(m)/P(p)/P(q)$  differentiable at  $f(a)$  in  $P(m)$  such that their composition  $f(h(x)) : [a, b]_m \rightarrow R(m, n)$  defined by  $oor = [oof(oooh([x]_p))]_m$  (and  $o, r$  defined in the obvious way), is also a digital function with values in  $P(m)$  (in other words its diagram commutes), and the product  $(df/dx) * (dh/dx)$  exists in  $P(m)$  too. Then their composition function is also a (digitally)  $P(m)/P(n)/P(q)$  differentiable function at  $a$  and

$$\frac{d(f(h(x)))}{dx} \Big|_a = \frac{df(y)}{dy} \Big|_{f(a)} * \frac{dh(x)}{dx} \Big|_a$$

Or in other symbols if  $df(a) = db$ ,  $df(h(a)) = d\gamma$ ,  $da = dx|_a$

$$\frac{d\gamma}{da} = \frac{d\gamma}{db} * \frac{db}{da}$$

**Hint for a proof:** We start with a seemingly infinitesimal  $dx$  of  $P(q)$  relative to  $P(p)$ , then from the  $P(m)/P(n)/P(p)$  differentiability of  $h$ , the  $dh(x)$  is a seemingly infinitesimal of  $P(p)$ ,

relative to  $P(n)$  and the derivative  $dh(x)/dx$  exists in  $P(m)$ . Taking this  $dh(x)$  seemingly infinitesimal of  $P(n)$  relative to  $P(m)$ , from the  $P(m)/P(n)/P(p)$  differentiability of  $f$ , the  $df(h(x))/dh(x)$  exists as element of  $P(m)$  and thus by multiplying  $(df(h(x))/dh(x)) * dh(x)/dx =_m df(h(x))/dx$ , the quotient by the hypotheses exists in  $P(m)$  therefore the composite is  $P(m)/P(n)/P(q)$  differentiable and the chain rule holds. QED

**PROPOSITION 2.8 (Linear combination)**

Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(p)/P(q)$  differentiable at a point  $x$  such that their linear combination  $adf(x)/dx + bdh(x)/dx$  for constants  $a, b$  of  $P(m)$ , is again inside  $P(m)$ . Then their linear combination  $af(x) + bh(x)$  function at  $x$  is also a (digitally)  $P(m)/P(n)/P(q)$  differentiable function and

$$\frac{d(af(x) + bh(x))}{dx} =_m a \frac{df(x)}{dx} + b \frac{dh(x)}{dx}$$

**Hint for a proof:** If  $dx$  is a seemingly infinitesimal of  $P(q)$  relative to  $P(p)$ , then it holds that  $d(af(x) + bh(x)) =_n a df(x) + b dh(x)$  is seemingly infinitesimal of  $P(p)$  relative to  $P(n)$ . Thus from the  $P(m)/P(p)/P(q)$  differentiability of the  $f$  and  $h$ , the  $d(af(x) + bh(x))/dx =_m a df(x)/dx + b dh(x)/dx$  is by hypotheses in  $P(m)$  too, and the property holds. QED

**PROPOSITION 2.9 (Leibniz product rule)**

Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(p)/P(q)$  differentiable at a point  $x$  such that the expression  $(df(x)/dx) * h(x) + f(x) * (dh(x)/dx)$  is again inside  $P(m)$ . Then the product  $f(x) * h(x)$  function at  $x$  is also a (digitally)  $P(m)/P(n)/P(p)$  differentiable function and

$$\frac{d(f * g)(x)}{dx} =_m \frac{df(x)}{dx} * h(x) + f(x) * \frac{dh(x)}{dx}$$

**Hint for a proof:** If  $dx$  is a seemingly infinitesimal of  $P(q)$  relative to  $P(p)$ , then  $d(f(x) * h(x)) =_p (f(x+dx)h(x+dx) - f(x)h(x)) =_p ((f(x)+df)(h(x)+dh) - f(x)h(x)) =_p (fdf+hdf+dfd)$  and by the ideal-like property of the infinitesimals it is in  $P(p)$ . Thus  $d(f(x)h(x))/dx =_m f(dh(x)/dx) + h(df(x)/dx) + df * (dh(x)/dx)$ . The last terms is zero in  $P(m)$  because the  $df$  is seemingly infinitesimal relative to  $P(m)$ , and the sum of the first two terms exists in  $P(m)$  by the hypotheses, thus the product is  $P(m)/P(n)/P(q)$  differentiable and the Leibniz product rule holds. QED

**PROPOSITION 2.8 (Quotient)**

Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(p)/P(q)$  differentiable at a point  $x$  such that their quotient  $f(x)/h(x)$  is defiable and in  $P(m)$  and the right hand of the formula below is computably finite, that is it belongs to  $P(m)$  when the terms of do. Then the quotient  $f(x)/h(x)$  function at  $x$  is also a (digitally)  $P(m)/P(n)/P(p)$  differentiable function and

$$d \frac{f}{h}(x) =_m \frac{\left(\frac{df(x)}{dx} * h(x) - f(x) * \frac{dh(x)}{dx}\right)}{[h(x)]^2}$$

**Hint for a proof:** Similar, as in the product rule. It is based on the ideal-like properties of the seemingly infinitesimals, and the hypotheses of the theorem. We start with a seemingly infinitesimal of  $P(q)$  relative to  $P(p)$ , and calculate the  $d(f/h)$ . We substitute the  $f(x+dx)$ ,  $h(x+dx)$  with  $f(x)+df$ ,  $h(x)+dh$  in  $P(p)$ , make the operations, we use the  $P(m)/P(p)/P(q)$  differentiability of the  $f$  and  $h$ , and that the right hand side of the formula in the theorem, also belongs to  $P(m)$  and we get the  $P(m)/P(n)/P(q)$  differentiability of the quotient. QED

**PROPOSITION 2.10 (Continuity of differentiable function)**

Let a digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , which is (digitally)  $P(m)/P(n)/P(p)$  differentiable at a point  $a$  of  $P(m)$ . Then it holds that it is also a (digitally)  $P(m)/P(n)/P(p)$  continuous function at  $a$ .

**Hint for a proof:** From  $f(x)' =_m df(x)/dx$  in  $P(m)$  and a seemingly infinitesimal  $dx$  of  $P(p)$  we get that  $df =_n f(x)' * dx$ . And from the ideal-like properties of the seemingly infinitesimals, the right hand side is also in  $P(n)$  and seemingly infinitesimal. Thus the  $f$  by the definition of continuity is  $P(m)/P(n)/P(p)$  digitally continuous. QED

**DEFINITION 2.4**

**(Higher dimension total derivative of a digital k-vector function.)**

Let  $A_m$  closed rectangle subset of  $P^k(m)$  and let a digital vector function  $f:A_m \rightarrow P^s(m)$ ,  $of: A_n \rightarrow P^s(n)$ ,  $f=[of]_m$ . We define that  $f$  is (digitally)  **$P(m)/P(n)/P(p)$  differentiable** at a point  $a$  in  $A_m$  iff there is a linear transformation  $L: P^k(m) \rightarrow P^s(m)$ , such that for any seemingly infinitesimal vector  $dh$  of  $P^k(p)$  relative to  $P^k(m)$ , it holds that

$$\frac{\|f(z+dh) - f(a) - L(dh)\|}{\|dh\|} =_m 0 \text{ in } P(m)$$

The linear transformation  $L$  is denoted by  $D(f(a))$  and is called total derivative of  $f$  at  $a$ . It can be proved that any such linear transformation  $L$  if it exists it is unique.

This is somehow equivalent to that

- 1) For every seemingly infinitesimal  $dh$  of  $P(p)^k$  at a point  $a$  of  $P(m)$ ,  $d_h f(a) =_m L(dh)$



2) And also for this seemingly infinitesimal  $dh$ , the  $d_h f(a) - L(dh)$  as seemingly infinitesimal of  $P(n)^k$  is transcendently smaller than the seemingly infinitesimal  $dh$  of  $P(p)^k$ .

$L$  can be a function of  $P(n)$  not only of  $P(m)$  that is definable in seemingly infinitesimals too.

Properties of classical total derivative are:

- 1) Partial derivatives per coordinate exist and their Jacobean matrix is the matrix of the total derivative (differential)
- 2) Conversely if they exist and are continuous in a region then the total derivative exist, and the digital vector function is called continuously differentiable.

### 3. THE DEFINITION OF THE DIGITAL ARCHIMEDEAN MEASURE AND INTEGRAL.

At first we define the digital Archimedean Integral and then also the Archimedean measure, although it can be vice versa.

#### DEFINITION 3.1

*Let a subset  $A$  of a closed interval  $[a,b]_n$  of  $P(n)$ , with  $[a,b]_m$  belonging to  $P(m)$ , of cardinal number of points  $|A|$  which is a number of  $P(n)$  and in general seemingly infinite relative to  $P(m)$ . We define as Archimedean measure of  $A$ , in symbols  $m(A)$ , and call  $A$ ,  $P(m)/P(n)$ -countably measurable, or simply  $P(m)/P(n)$ -measurable, a possibly of seemingly infinite terms relative to  $P(m)$  sum of  $|A|$  times of the  $P(n)$ -sizes of the points of  $A$ , such that the  $P(m)$  rounding of the sum belongs to  $P(m)$ . In other words as each point of  $A$  in  $P(n)$  has size  $10^{-n}$  then  $m(A) = \lfloor |A| * 10^{-n} \rfloor_m$  which is a number required to belonging to  $P(m)$  for  $A$  to be  $P(m)/P(n)$ -measurable.*

**Similar definition exists for higher dimensions  $R^k(m,n,p,q)$**

#### DEFINITION 3.2

*Let a digital functions  $f: [a,b]_m \rightarrow P(m)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f = [of]$ . Then we define as Archimedean  $P(m)/P(n)$ -integral of  $f$  on the closed interval  $[a,b]_m$ , and call the  $f$  Archimedean  $P(m)/P(n)$ -integrable, the possibly of seemingly infinite terms relative to  $P(m)$  sum of  $\lfloor [a,b]_n \rfloor$  times of the  $P(n)$ -sizes of the points  $dx$  of  $[a,b]_n$  multiplied with the value of  $f(x)$  at each point  $dx$  of  $[a,b]_n$ , such that the  $P(m)$  rounding of this weighted sum belongs to  $P(m)$ . In symbols*

$$I = {}_m \int_a^b f(x) dx \text{ in } P(m)$$

*Notice that according to that definition the Archimedean measure of a subset  $A$  of  $[a,b]_m$  is the Archimedean of the characteristic function  $X_A$  of  $A$ . In symbols*

$$m(A) =_m \int_a^b X_A dx \text{ is in } P(m)$$

**Similar definition exists for higher dimensions  $R^k(m,n,p,q)$**

**Similarly we may define measure and integration by the pairs of precision levels  $P(m)$ - $P(p)$ , and  $P(m)$ - $P(q)$  etc.**

The basic properties of the classical Integral are:

- 1) Continuous=> Integrable
- 2) Linearity
- 3) Inequality
- 4) Additivity at the limits of integration
- 5) Upper, Lower bounds and the limits of integration
- 6) Absolute value inequality
- 7) Additive property of point measure

$$m(A \cup B) =_m m(A) + m(B) - m(A \cap B)$$

- 8) It holds also that functions that differ only at a set of measure zero have the integrals.

**PROPOSITION 3.1(Measure zero)**

*Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(n)$  integrable on  $[a,b]_m$ , such that they differ in values only on a subset of  $[a,b]_m$  of (Archimedean) measure zero, then their (Archimedean) integrals are equal.*

$$\int_a^b f(x) dx =_m \int_a^b h(x) dx$$

**Hint for a proof:** It suffices to prove that the Integral of their difference is zero. Which is point-wise zero at all points of  $[a,b]_n$  in  $P(n)$ , except at the points of a subset A of the closed interval of measure zero,  $m(A) =_m 0$ . Since the A is a finite set, the  $f(A)$  has a maximum M in  $P(m)$ , which when factored out in the finite sum which is the Archimedean  $P(m)/P(n)$  integral, it will give an upper bound for the integral of the  $f(x)-h(x)$ , of the type  $M * m(A)$ . But as  $m(A)=0$ , then the integral of  $f(x)-h(x)$  is also zero in  $P(m)$  QED.

**PROPOSITION 3.2 (Continuity implies integrability)**

Let a digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , which is (digitally)  $P(m)/P(n)/P(p)$  continuous in the closed interval  $[a,b]$ . Then it holds that it is also a (digitally)  $P(m)/P(p)$  integrable function at  $[a,b]_m$  and

$$I =_m \int_a^b f(x)dx \text{ is in } P(m)$$

**Hint for a proof:** Since the  $f(x)$  is continuous on  $[a,b]$  by proposition 2.6, it has a maximum  $M$ . As in the proof of the previous proposition when  $M$  is factored out in the finite sum which is the Archimedean  $P(m)/P(n)$  integral, it will give an upper bound for the integral of the  $f(x)$ , of the type  $M*|b-a|$ . Therefore the integral sum is upper bounded in  $P(m)$  and it exists therefore as a number of  $P(m)$ . Thus  $f(x)$  is  $P(m)/P(n)$  integrable QED.

**PROPOSITION 3.3 (Additive decomposition of interval)**

Let a digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , which is (digitally)  $P(m)/P(n)$  integrable on the closed interval  $[a,b]_m$ . Then for an  $c$  of  $[a,b]_m$  in  $P(m)$  it holds that  $f$  it is also a (digitally)  $P(m)/P(n)$  integrable function on  $[a,c]_m$  and  $[c,b]_m$  and

$$\int_a^b f(x)dx =_m \int_a^c f(x)dx + \int_c^b f(x)dx$$

**Hint for a proof:** Direct consequence from the associative property of finite sums in  $P(n)$ . QED

**PROPOSITION 3.4 (Linear combination)**

Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(n)$  integrable on  $[a,b]_m$ , such that their linear combination  $kf(x)+lh(x)$  for constants  $k, l$  of  $P(m)$ , is again inside  $P(m)$ . Then their linear combination  $kf(x)+lh(x)$  function is also (digitally)  $P(m)/P(n)$  integrable digital function on  $[a,b]_m$  and

$$\int_a^b (kf(x) + lh(x))dx =_m k \int_a^b f(x)dx + l \int_a^b h(x)dx$$

**Hint for a proof:** Direct consequence from the associative and distributive law, of finite sums in  $P(n)$ . QED.

**PROPOSITION 3.5 (Upper, Lower bounds inequalities)**

Let a digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , which is (digitally)  $P(m)/P(n)$  integrable on the closed interval  $[a,b]_m$ , such that for constants  $m, M$  of  $P(m)$ , it holds that  $m \leq_m \int_a^b f(x) dx \leq_m M$ . Then

$$m^*(b-a) \leq_m \int_a^b f(x) dx \leq_m M(b-a).$$

**Hint for a proof:** Direct consequence from the distributive law and corresponding inequalities of finite sums. QED.

**PROPOSITION 3.6 (Integrability)**

Let a digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , which is upper bounded by a number of  $P(m)$ :  $f(x) \leq_m M$  and  $M$  and also  $(b-a)M$  are in  $P(m)$  for all  $x$  in  $P(n)$ . Then it is Archimedean  $P(m)/P(n)$  integrable:

$$I =_m \int_a^b f(x) dx \text{ exists as a number of } P(m)$$

**Indication for a Proof:** In the definition of the Archimedean integral, in the finite (but seemingly infinite) sum of terms  $f(x)dx$  in  $P(n)$  we may substitute  $f(x)$  with its bound  $M$ , and factor out the  $M$ , by the distributive law of finite sums, while the sum of  $dx$ 's give the length of the interval  $[a,b]_m =_m b-a$ . Therefore the integral is upper bounded by  $(b-a)M$  in  $P(m)$ , which means that the rounded in  $P(m)$  sum and Integral exists also in  $P(m)$ , thus the function is Archimedean integrable.

**PROPOSITION 3.7 (Inequality with absolute values)**

Let a digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$   $f:[a,b]_n \rightarrow P(m)$ , which is integrable on  $[a,b]_m$ . Then it holds that  $|f|$  is also integrable on  $[a,b]_m$  and

$$\left| \int_a^b f(x) dx \right| \leq_m \int_a^b |f(x)| dx$$

**Hint for a proof:** Direct consequence from the corresponding same inequality property of absolute values for finite sums in  $P(n)$ . QED

**PROPOSITION 3.8 (Integration by parts)**

Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(p)$  integrable on  $[a,b]_m$ , such that the next integrals on  $[a,b]_m$  exist

$\int_a^b f(x) \left(\frac{dh(x)}{dx}\right) dx, \int_a^b h(x) \left(\frac{df(x)}{dx}\right) dx$   
then

$$\int_a^b f(x) \left(\frac{dh(x)}{dx}\right) dx + \int_a^b h(x) \left(\frac{df(x)}{dx}\right) dx =_m f(b)h(b) - f(a)h(a)$$

Where the derivatives are  $P(m)/P(n)/P(p)$  differentiation.

**Hint for a proof:** Remember that here the seemingly infinitesimals  $dx$  are real finite numbers of  $P(p)$ . By cancelling out the  $dx$  in the integrals in the left side, and substituting the  $df(x)$ ,  $dh(x)$ , with their equals in  $P(p)$ ,  $f(x+dx)-f(x)$ ,  $h(x+dx)-h(x)$ , we multiply out them so that these terms as terms of successive finite differences in the finite sum, which is the integral cancel out, to give the right hand side. QED.

**PROPOSITION 3.9 (Inequality 2)**

Let two digital functions  $f:[a,b]_m \rightarrow R(m,n)$ , with  $of: [a,b]_n \rightarrow P(n)$ ,  $f=[of]$ , and  $h:[a,b]_m \rightarrow R(m,n)$ , with  $oh: [a,b]_n \rightarrow P(n)$ ,  $h=[oh]$ , that are (digitally)  $P(m)/P(n)$  integrable on  $[a,b]_m$  and  $f(x) \leq_n h(x)$  in  $[a,b]_n$  then it holds that

$$\int_a^b f(x) dx \leq_m \int_a^b h(x) dx$$

**Hint for a proof:** Direct consequence from the corresponding similar property of finite sums, which is the integral here. QED.

**PROPOSITION 3.10 (Additivity of Archimedean measure )**

Let a sets  $A, B$ , in  $P(n)$  that are Archimedean measurable. Then also their union  $A \cup B$  and their intersection  $A \cap B$  are Archimedean measurable and it holds for their Archimedean measure symbolized by  $m()$ , that

$$m(A \cup B) =_m m(A) + m(B) - m(A \cap B).$$

**Hint for a proof:** Direct consequence from the corresponding same formula of cardinality of finite sets, and the definition of the Archimedean  $P(m)/P(n)$  measure of a set as finite sum of that of its points. QED.

**Fubini Theorem** It can be deduced as in classical Calculus that we can get the value of the integral by iterative one dimensional integrals once the lower or upper one-dimensional integrals exist. It is the results Associative and commutative property of finite sums.

**PROPOSITION 3. 12 (Fubini theorem iterated integrals)**

Let  $A$  closed rectangle subset of  $P^k(m)$  and  $B_m$  closed rectangle subset of  $P^s(m)$  and let digital function  $f:AxB \rightarrow P(m)$ ,  $of: AxB \rightarrow P(n)$ ,  $f=[of]$  (digitally) integrable.

For  $x$  in  $A$  let  $h_x : B \rightarrow P(m)$  be defined by  $h_x(y)=_mf(x,y)$ , and we assume that it is also a digital function and let

$$I(x) =_m \int_B (y)dy$$

which is assumed also a digital function.

Then  $I(x)$  is (digitally) integrable on  $A$  and it holds that

$$\iint_{AxB} f(x, y) dydx =_m \int_A I(x)dx =_m \int_A \left( \int_B h_x(y)dy \right) dx$$

**Hint for a proof:** Direct consequence of the associative and distributive property of the finite sums. QED

**4. THE FUNDAMENTAL THEOREM OF THE DIGITAL CALCULUS AS THE RELATION OF THE DIGITAL DERIVATIVE WITH THE DIGITAL INTEGRAL.**

It is simply the formal expression that a weighted sum that is the mass of segment when getting its derivative to length it will give the linear density of the segment, which is also a derivative.

**PROPOSITION 4.1 (FUNADMENTAL THEOREM OF CALCULUS)**

Let a digital functions  $f:[a,b]_m \rightarrow P(m)$ , with  $f=[of]$

$f:[,ab]_n \rightarrow P(n)$ , which is (digitally)  $P(m)/P(n)/P(p)$  continuous thus  $P(m)/P(p)$  integrable on the closed interval  $[a,b]_m$  and also the next function on  $[a,b]_n$  is a digital function.

$$h(x) =_n \int_a^x f(y)dy$$

Then it holds that the function at  $[a,b]_m$

$$h(x) =_m \int_a^x f(y)dy$$

is (digitally)  $P(m)/P(n)/P(p)$  differentiable and at any  $c$  of  $[a,b]_m$  .

$$\left. \frac{dh(x)}{dx} \right|_c =_m f(c)$$

**Hint for a proof:** For a seemingly infinitesimal  $dx$  of  $P(p)$  relative to  $P(n)$  the  $dh(x)=_n h(x+dx)-h(x)$ . But by the proposition 3.3 of additive decomposition of the integral over its intervals of integration gives  $h(x+dx)-h(x)=_n h(x)+f(x)dx-h(x)$ , thus  $dh(x)=_n f(x)dx$ . And by the ideal-like properties of the seemingly infinitesimals it is also a seemingly infinitesimal of  $P(n)$ . Thus the  $h$  is  $P(m)/P(n)/P(p)$  differentiable with derivative equal to  $f(x)$  in  $P(m)$ . QED.

## 5. CONCLUSIONS AND PERSPECTIVES.

For all practical reasons in the physical and social sciences the digital calculus gives all the well-known applications with a finite ontology which is directly realizable both in the physical ontology of atomic matter or digital ontology of operating systems of computers. This has vast advantages in applications in, Engineering, Physics, Meteorology, Chemistry, Ecology, social sciences etc.

The digital Calculus also resurrects the 17<sup>th</sup> and 18<sup>th</sup> century mathematical arguments in Calculus and mathematical physics that treated the “infinitesimals” as separate entities in the derivatives.

The digital Calculus is also an educational revolution in the Education of Mathematics. . It is a new method of teaching mathematics where there is higher integrity with what we say, write, see, and think.

After [8] that defines the axiomatic Euclidean geometry and the current outline of the digital Differential and Integral Calculus, one may define and solve the digital differential and partial differential equations as essentially difference equations, (with easier applications in the physical sciences), digital fluid dynamics (with easier applications in physics), digital differential geometry, digital functional analysis (appropriate for easier applications in signal theory) etc. The road is open and the digital world of the computers is the direct tool for this.

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# **THE FICTIONAL DIALOGUE OF THE IMMORTAL MATHEMATICIANS ON THE OCCASION OF THE NEW DIGITAL DIFFERENTIAL AND INTEGRAL CALCULUS**

**ARXHINEOMEDES** after presenting the immortals the basic introduction to the digital differential and integral calculus , invites them in a free discussion about it.

**ARCHINEOMEDES**, and **NEWCLID** are individuals representing the collective intelligence of the digital technology but also of mathematics of the 21<sup>st</sup> century.

The participants of the discussion are the next 20.

- 1. Pythagoras**
- 2. Eudoxus**
- 3. Euclid**
- 4. Democritus**
- 5. Archimedes**
- 6. Newton**
- 7. Leibnitz**
- 8. Cartesius**
- 9. Cauchy**
- 10. Dedekind**
- 11. Weierstrasse**
- 12. Hilbert**
- 13. Riemann**
- 14. Cantor**
- 15. von Neumann**
- 16. Poincare**
- 17. Gödel**
- 18. Cavalieri**
- 19. Lagrange**
- 20. Helmholtz**
- 21. Gauss**
- 22. Jordan**
- 23. Lobachevski**
- 24. Zeno**
- 25. Bolzano**
- 26. Lebesgue**

**And 2 contemporary mortals:**  
**20. Newclid, and Archineomedes.**

**ARCHINEOMEDES:**

**Welcome honourable friends that you have become immortals with your fame and contribution in the creation of the science and discipline of Mathematics among the centuries on the planet earth!**

Now that you have watched my presentation of the digital differential and integral calculus , I would like to initiate a discussion that will involve your remarks , perspectives opinions about it. Who would like to start the conversation?

**PYTHAGORAS:**

Thank you Archineomedes for the honor in gathering us together. I must express that I like the new of the digital differential calculus as well as the approach of the Axiomatic Digital Euclidean Geometry of Newclid , that as you say is a resume of what already the beginning of the 21<sup>st</sup> century in the earthly Computer Science has realized through software in the computer operating systems and computer screens and monitors.

In fact, I was always teaching my students that the integer natural numbers are adequate for creating a mathematical theory of the geometric space! And this is so because matter has atomic structure as Democritus has taught and space and time are simply abstract properties of matter. One only has e.g. to take as unit of measurement of lengths, the length of a invisible points and all metric relations in the low precision level of the figures, including the Pythagorean theorem, become relations of positive integer numbers, or solutions of Diophantine equations! But at that time no such detailed and elaborate system of definitions was easy, neither a well accepted concept that matters consists from atoms, was available in the mathematicians of the ancient Greece, Egypt or Babylon.

**EUCLID:** I am impressed ARCHINEOMEDES with your skillful definitions of derivative and integral. To tell the truth I never was satisfied with the classical definitions through limits and the infinite which now I consider a phenomenological abstraction not so much appropriate for an ontology of mathematics with applications in the physical sciences. I am myself also indirectly responsible for it, as my axioms that for every two points on line there is always a 3<sup>rd</sup> between them, was the beginning of the need for the infinite and I am glad now that we can do the mathematics without it.

**ARCHIMEDES:** I like your digital differential and integral calculus Archineomedes ! It is as my perceptions! Actually my heuristic experimental work with solids that I was filling with sand or water to make volume comparisons by mechanical balances, is just an experimental realization of your concept of measure and integral through those of the points and finite many points! That is how I discovered and proved the formula of the volumes of the sphere.

**ARCHINEOMEDES:** Thank you Archimedes, that is why for your honor I called them Archimedean point measures.

**DEMOCRITUS:** Bravo ARCHINEOMEDES! Exactly my ideas of atoms! Actually as in my theory of atoms, the water is made from finite many atoms, the volume experiments of Archimedes with water is rather the exact realization of your point measures for areas and volumes through that of the invisible points! Here the atoms of the water are invisible, while the granulation of the sand may resemble your concept of the visible points!

**LEIBNITZ:** I want to congratulate you ARCHINEOMEDES for your approach! In fact my symbols of infinitesimal  $dx$  in my differential calculus suggest what I had in mind: A difference  $dx=x_2-x_1$  so that it is small enough to be zero in the Lowest phenomenological measurements precision level but still non-zero in the Highest ontological precision level! Certainly a finite number!

**NEWTON:** I must say here that the Leibnitz idea of infinitesimal as a finite number based on the concepts of Low and High precision is not what I had in my mind when I was writing about infinitesimals or fluxes. That is why I was calling them fluxes and symbolized them differently. The theory of null sequences of numbers (converging to zero) of Cauchy and Weierstrasse is I think the correct formulation of my fluxes. Nevertheless these null sequences need not be infinite, they can very well be finite ending on the finest bin of the highest precision level. I was believing in my time ,like Democritus, that matter consist from finite many atoms , but I never dared to make a public scientific claim of it, as no easy proof would convince the scientist of my time!

I want to ask an important question to ARCHINEOMEDES : Is your digital differential and integral calculus based on three levels of precision more difficult or simpler that the classical differential and integral calculus based on limits and infinite many real numbers? And could be formulated in a equivalent way with fluxes, that is finite sequences converging to a point?

**ARCHINEOMEDES:** Well Newton thank you for the good words! A differential and integral calculus based on three levels of precision is certainly less complicated than ( and also not equivalent to) the classical calculus with infinite sequences or limits. But a differential and Integral calculus of 3 ,4 or more precision levels is by far more complicated than the classical analogue differential and Integral calculus. Only that this further complication is a complexity that does correspond to the complexity of the physical material reality, while the complexities of the infinite differential and integral calculus (in say Lebesgue integration theory or bounded variation functions etc) is a complexity rather irrelevant to the physical material complexity. Now I do believe , although I have not carried it out with proofs, that by using finite sequences converging to a point of the highest precision level, as seemingly infinitesimals, would be an equivalent formulation for the digital derivative.

**CARTESIUS:** I want to congratulate you ARCHINEOMEDES for your practical , finite but comprehensive digital differential and integral calculus which is practically based on the digital analytic Cartesian geometry! And my arithmetization of the geometry is the prerequisite for a digitalization. This was my implicated intention too.

**CAUCHY:** I wish I had thought of such definitions of the real numbers and integral myself, including the concept of seemingly. I want nevertheless to ask a very important question ARCHINEOMEDES. You mentioned that the digital differential and integral calculus is not equivalent to the classical differential and integral calculus with the infinite. Is it possible in the context of your concepts to define a differential and integral calculus equivalent to the classical one?

**ARCHINEOMEDES:** Well CAUCHY I have thought about it, although I never carried out detailed proofs. It seems to me that if I take all possible, I mean all levels of fine sizes of precision levels and require that a function would be digital differentiable or digitally integrable in all of them , then this might be equivalent to the classical definitions. Nevertheless such a very strong requirement would be, absolute, and not corresponding to the situations of material ontology. It would be a very strong requirement tying strongly together the phenomenology and ontology of matter, and we do know, that they should differ.

**EUDOXOS:** Well in your digital real numbers ARCHINEOMEDES , my definition of the ratio of two linear segments which is the base of the complete continuity of the line seem not to be that critical in your system, although I think that it still holds, no?

**ARCHINEOMEDES:** It still holds EUDOXOS, except it is restricted to rational numbers with finite decimal representation.

**DEDEKIND:** And as I reformulated the idea and definition of equality of ratio of linear segments of Eudoxus, to my concept of Dedekind cuts about the completeness of continuity of the real numbers, does this still hold in your digital real numbers?

**ARCHINEOMEDES:** It still holds DEDEKIND, except it is restricted to digital numbers with finite decimal representation. And the same with the supremum and infimum properties of bounded sets of real numbers.

**BOLZANO:** That is why my basic theorem of continuous curves of continuous functions holds in the digital calculus. As ARCHINEOMEDES presented we have a usual and classical topological space based in this continuity.

**JORDAN:** Which suggests also that my theorem of closed curves in the digital plane should hold too?

**ARCHINEOMEDES:** Certainly JORDAN although I have not carried out any detailed proof of it yet.

**WEIESSTRASSE:** What about my definition of continuity with the epsilon-delta inequalities, does it hold for the digital continuity ARCHINEOMEDES?

**ARCHINEOMEDES:** More or less yes, with slightly different details WEIESSTRASSE. The epsilon must be restricted to the lowest phenomenological precision level, while the delta in the highest ontological precision level. That is how I thought initially to define the digital continuity, but later I preferred the concept of seemingly infinitesimal so as to resurrect as rigorous and correct the arguments of hundreds of mathematicians in the 7th, 18th and 19th century in the calculus that used infinitesimals with the Leibniz notation.

**POINCARÉ:** Yes indeed, my articles in mathematics are full of arguments using the infinitesimals in an isolated way. Thank you ARCHINEOMEDES that now they have a rigorous and exact, formulation within the finite. I used to mock those that made mathematics with transfinite numbers, but now with the digital real numbers I realize that the inverse of a seemingly infinitesimal is a seemingly infinite number. I used to say that mathematics and geometry is the art of correct reasoning over not-corresponding and incorrect figures. With the digital mathematics this is corrected.

**CAVALIERI:** Would the methods of digital calculus render my principle of indivisibles on the calculations of volumes of 3-dimensional bodies rigorous and exact too?

**ARCHINEOMEDES:** With the right new details I believe yes CAVALIERI. A slice of a 3-D body by a digital plane, would consist from finite many highest precision level invisible points as tiny cubes that still have finite thickness (although zero in the

phenomenological lowest precision level) therefor they make a kind of indivisibles and indivisible slices.

**LEBESQUE:** So in your digital calculus ARCHINEOMEDES, the definition of the digital integral is some how a Lebesgue integral or a Riemann integral?

**ARCHINEOMEDES:** I did not define the integral with partitions to answer it precisely. If would do so, then in your integral could start with seemingly infinite partitions, while a finite Riemann integral with only computable finite partitions. I defined it directly with seemingly infinite many, seemingly infinitesimal rectangles. So it is closer to your definition, and its relation with digital functions with points of discontinuity of measure seemingly zero rather confirms it.

**HILBERT:** I like your brave and perfect approach ARCHINEOMEDES! No infinite in your calculus till very realistic and useful, so as to have easy physical applications, as nothing in the physical material reality is infinite. Congratulations!

**Von NEUMANN:** I like too your digital differential and integral calculus ARCHINEOMEDES! I believe that I could easily make it myself, except at that time I was busy in designing a whole generation of computers! I believe your work is a direct descendant of my work on computers. As you said your ideas came from software developers in the operating system of a computer!

**ARCHINEOMEDES:** Indeed, von Neumann! Thank you!

**CANTOR:** Pretty interesting your digital differential and integral calculus ARCHINEOMEDES! But what is wrong with the infinite? Why you do not allow it in your mathematic? I believe that the infinite is a legitimate creation of the human mind! Your Digital differential and integral calculus lacks the charm and magic of the infinite!

**PYTHAGORAS:** Let me, ARCHINEOMEDES, answer this question of CANTOR! Indeed CANTOR the human mind may formulate with a consistent axiomatic way what it wants! E.g. an axiomatic theory of the sets where infinite sets exist! And no doubt that the infinite is a valuable and sweet experience of the human consciousness! But as in the physical material reality there is nowhere infinite many atoms, mathematical models that in their ontology do not involve the infinite, will be more successful for physical applications! In addition, there will not be any irrelevant to the physical reality complexity as in the mathematical models of e.g. of physical fluids that use infinite many points with zero dimensions in the place of the finite many only physical atoms with finite dimensions. The infinite may have its charm. Actually I believe that the human consciousness gives the feeling of the infinite. Consciousness is not a property of matter like energy, and it has to remain outside the ontology of matter and of mathematics. The Digital Differential and Integral Calculus has its own and different magic too!

**RIEMANN:** Very impressive ARCHINEOMEDES ,your phenomenological-ontological and logical approach to the Differential and Integral! But what about my Riemannian geometric spaces? Could they be formulated also with Local, Low and High precision levels and finite many visible and invisible points?

**ARCHINEOMEDES:** Thank you Riemann! Well my friend any digital system of your Riemannian Geometric spaces, with finite many points might require more than 2 probably 3 or 4 precision levels! That is why I start with a system of digital numbers of 4 precision levels. The reason might be that at any A point of a Riemannian Space, the tangent or infinitesimal space at A is Euclidean! And here the interior of the point A will be a whole flat Euclidean space which already requires two or 3 precision levels and both the visible and invisible points of the tangent Euclidean space will have to be invisible, while the point A visible point! But let us have patience! In the future I will study and answer your question with details and clarity! Originally me and Newclid had defined the digital real numbers only with two precision levels. But for the sake of differential manifolds and your Riemannian geometry I decided to put in the definition 4 precision levels. In modern software technology e.g. in scalable software maps, there are many map scales or precision levels that might be involved.

**LOBACHEVSKI:** I assume that the digitalization of the Riemannian geometry will derive automatically a digital version of my Hyperbolic non-Euclidean 3-dimensional geometry too!

**ARCHINEOMEDES:** Certainly LOBACHEVSKI!

**HELMHOLTZ:** I think that the idea of digital space and calculus is closer to the physical reality. Even my theory and study of sound, when stepping down to the molecules and atoms of air and matter becomes a digital ontology.

**LAGRANGE:** Of course HELMHOLTZ! As Cartesius arithmetization of geometry by coordinates is a prerequisite for the digitalization of geometry, so my arithmetization of the physical magnitudes of motion like velocity , force acceleration etc is a prerequisite for the realistic digitalization of such physical magnitudes of motion through the digital real numbers and digital derivative and integral.

**ZENO:** So if the magnitudes of motion after Lagrange arithmetization, are now digital in your calculus ARCHINEOMEDES, would this mean that my paradox with Achilles and the turtle resolve differently?

**ARCHINEOMEDES:** Certainty Zeno. In the classical “analogue” mathematics of the infinite, your paradox is resolved , as the sum of infinite series which is nevertheless a finite number. In the digital calculus, the corresponding series is already a finite series (as the space and magnitudes motion are themselves digital and finite) and it has also a computable finite number as its sum.

**GAUSS:** I agree that the digital differential and integral calculus is more transparent lucid and practical. What about my different proofs of the fundamental theorem of algebra that any polynomial has at least one root in the complex numbers. Do you think

**ARCHINEOMEDES** that they could be transferable to proofs in the digital complex numbers?

**ARCHINEOMEDES:** Although I have not carried out in detail any such transfer of your alternative proofs, I believe it can be done. The digital ontology as Newclid had mentioned in previous discussions allows also for a new type of proofs which is that of finite induction on the (finite many) points. Maybe still an new alternative proof can be obtained in this way.

**GOEDEL:** You and **NEWCLID** , **ARCHINEOMEDES** mentioned somewhere that all of your arguments take place in the digital logic. What is the difference of digital logic say compared to a 1<sup>st</sup> order formal logic of classical mathematics?

**ARCHINEOMEDES:** The main difference **GOEDEL** is that in the digital 1<sup>st</sup> order formal Logic there do not exist countably infinite many proofs, or countable many formulae. Only finite many up to some size, as the digital natural numbers are used and not the classical natural numbers of Peano axioms.

**GOEDEL:** This mean that my theorem that for every axiomatic theory that contains the natural numbers it exist at least one proposition **A** than neither **A**, neither the negation of **A** can be proved, does not exist in your digital meta-mathematics.

**ARCHINEOMEDES:** That is correct **GOEDEL**.

**GOEDEL:** So in your digital mathematics, there might exist at least one axiomatic theory **T**, that contains the digital natural numbers **N** , and within a digital logic **L**, so that for every proposition **A**, there is an appropriate size digital logic **L(A)**, such that there is a proof either of **A** or of the negation of **A**?

**ARCHINEOMEDES:** It is rather a more optimist theorem for the powers of rational thinking this theorem, compared to your celebrated theorem **GOEDEL** is it not? Well I have not laid down the details of its statement and the details of any proof of it, but I certainly hope that it might hold true.

**NEWCLID:** I want also to ask you **ARCHINEOMEDES** to make it clear if your concepts of digital line and digital plane and visible geometric points on them are different compared to those in my axiomatic system of Digital (but continuous) Euclidean geometry.

**ARCHINEOMEDES:** There are certainly **NEWCLID** considerable difference. Actually I did not have to define, what a digital line or digital plane is. But if I would have to, I would use linear equations of analytic geometry with rounding in the precision levels. And in my case the visible points are clearly tiny cubes. In your axiomatic system you have as initial concept the linear segment and plane and visible and invisible points , and you impose axiomatically mutual properties of them. As I understand your axiomatic system it is open at each instance how the visible points are chosen say for a linear segment. Maybe there are more than one possible choices each time of visible points that satisfies the axioms. The very inability to have simultaneously that the coordinates are in 1-1 correspondence with finite many points and that congruence is also a 1-1 correspondence also of the finite many points was the source of incommensurable magnitudes and irrational numbers is he ancient Greece. And although you impose

**axiomatically that each point has coordinated and each coordinate is coordinate of some point, nowhere in your axioms there as the requirement of an 1-1 such correspondence but only of a many to many relation. In my case a prefer to have a 1-1 correspondence of coordinates with points and avoid any claims of a congruence relation of figures that I never define for the needs of differentiation and integration. In addition you use a 3-precision levels system of digital real numbers with one resolution of visible and 2 resolutions of invisible points and I prefer to use a 4-precision levels system of digital real numbers with 2 resolutions of visible points and 2 resolutions of invisible points.**

**ARCHINEOMEDES: If there no more questions or remarks, let us end here our discussion, and let us take a nice and energizing walk under the trees in the park close to Plato's academy.**

**AT THIS POINT THE DISCUSSION ENDS.**